2550

HW 3

Solutions

1 (a)

This is already a 1

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right) \xrightarrow{R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right)$$

make these zeros

$$\xrightarrow{-3R_1+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & \boxed{-1} & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right)$$

now make this into a 1

$$\xrightarrow{-R_2\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & \boxed{-10} & -2 & -14 \end{array}\right)$$

make this a zero

$$\xrightarrow{10R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & \boxed{-52} & -104 \end{array}\right)$$

make this into a 1

$$\xrightarrow{-\frac{1}{52}R_3\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The reduced system is:

$$\boxed{\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ x_2 - 5x_3 &= -9 \\ x_3 &= 2 \end{aligned}}$$

leading variables are x_1, x_2, x_3.

No free variables.

$$\boxed{\begin{aligned} x_1 &= 8 - x_2 - 2x_3 & (1) \\ x_2 &= -9 + 5x_3 & (2) \\ x_3 &= 2 & (3) \end{aligned}}$$

$\Rightarrow$

Solve (3):

$x_3 = 2$

Plug into (2):

$x_2 = -9 + 5x_3$
$= -9 + 5(2) = 1$

Plug into (3):

$x_1 = 8 - x_2 - 2x_3$
$= 8 - 1 - 2(2)$
$= 3$

Thus, the system has only one solution.

$x_1 = 3,\ x_2 = 1,\ x_3 = 2$

①(b)

make this into a 1

$$\left(\begin{array}{ccc|c} \boxed{2} & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ \boxed{-2} & 5 & 2 & 1 \\ \boxed{8} & 1 & 4 & -1 \end{array}\right)$$

make these zeros

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right)$$

$$\xrightarrow{-8R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & \boxed{7} & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right)$$

make this into a 1

$$\xrightarrow{\frac{1}{7}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & \boxed{-7} & -4 & -1 \end{array}\right)$$

make this a zero

$$\xrightarrow{7R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The reduced system is:

$$\boxed{x_1} + x_2 + x_3 = 0$$
$$\boxed{x_2} + \frac{4}{7}x_3 = \frac{1}{7}$$
$$0 = 0$$

leading variables are x_1, x_2

Free variable is x_3

$$x_1 = -x_2 - x_3 \quad (1)$$
$$x_2 = \frac{1}{7} - \frac{4}{7}x_3 \quad (2)$$

Set free variables:

$x_3 = t$

Solve (2):

$x_2 = \frac{1}{7} - \frac{4}{7}x_3 = \frac{1}{7} - \frac{4}{7}t$

Solve (1):

$x_1 = -x_2 - x_3 = -\left(\frac{1}{7} - \frac{4}{7}t\right) - t = -\frac{1}{7} - \frac{3}{7}t$

Solution:

$$x_1 = -\frac{1}{7} - \frac{3}{7}t$$
$$x_2 = \frac{1}{7} - \frac{4}{7}t$$
$$x_3 = t$$

where t is any real number

See next page

So, there are an infinite number of solutions, one for each t.

For example, if $t=0$ then

$$\boxed{\begin{aligned} x_1 &= -\frac{1}{7} - \frac{3}{7}(0) = -\frac{1}{7} \\ x_2 &= \frac{1}{7} - \frac{4}{7}(0) = \frac{1}{7} \\ x_3 &= 0 \end{aligned}}$$

is a solution.

Or if $t=1$, then

$$\boxed{\begin{aligned} x_1 &= -\frac{1}{7} - \frac{3}{7}(1) = -\frac{4}{7} \\ x_2 &= \frac{1}{7} - \frac{4}{7}(1) = -\frac{3}{7} \\ x_3 &= 1 \end{aligned}}$$

is another solution.

And so on.

1 (c)

we already have a 1 here

$$\left(\begin{array}{cccc|c} \boxed{1} & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right)$$

make these into zeros

$$\xrightarrow[-3R_1+R_4\to R_4]{\substack{-2R_1+R_2\to R_2 \\ R_1+R_3\to R_3}} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & \boxed{3} & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right)$$

make this a 1

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right)$$

make these into zeros

$$\xrightarrow[-3R_2+R_4 \to R_4]{-3R_2+R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The reduced system is:

$$\boxed{\begin{aligned} x - y + 2z - w &= -1 \\ y - 2z &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

leading variables are x, y

Free variables are z, w

$$\boxed{\begin{aligned} x &= -1 + y - 2z + w \quad ① \\ y &= 2z \quad ② \end{aligned}}$$

Set free variables: $z = t$, $w = s$

Solve ②: $y = 2z = 2t$

Solve ①: $x = -1 + y - 2z + w = -1 + 2t - 2t + s$
$= -1 + s$

Solution:

$$x = -1 + s$$
$$y = 2t$$
$$z = t$$
$$w = s$$

where t, s are any real numbers

There are an infinite number of solutions.
For example, setting $s = 0$ and $t = 0$ gives $x = 0, y = 0, z = 0, w = 0$.
Setting $s = 1, t = 2$ gives
$x = 0, y = 4, z = 2, w = 1$.

1(d)

Need to put a 1 up here

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

make those zeros

now make this a 1

$$\xrightarrow{-6R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 27/3 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & -6 & 9 & 27/3 \end{array}\right)$$

now make this a zero

$$\xrightarrow{6R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

The reduced system is:

$$\boxed{\begin{aligned} a + 2b - c &= -2/3 \\ b - \tfrac{3}{2}c &= -\tfrac{1}{2} \\ 0 &= 6 \end{aligned}}$$

Answer

Since we have $0 = 6$ there are no solutions; the system is inconsistent

② (a)

turn into a 1

$$\left(\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -3/2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right)$$

now make these zeros using the 1 in row 1

$$\xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -3/2 & -1 \\ 0 & 4 & 3 \\ 0 & 13/2 & 4 \end{array}\right)$$

now make this a 1

$$\xrightarrow{\frac{1}{4}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -3/2 & -1 \\ 0 & 1 & 3/4 \\ 0 & 13/2 & 4 \end{array}\right)$$

use the 1 in row 2 to make this a zero

$$\xrightarrow{-\frac{13}{2}R_2+R_3 \to R_3} \left(\begin{array}{cc|c} 1 & -3/2 & -1 \\ 0 & 1 & 3/4 \\ 0 & 0 & -7/8 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -3/2 & -1 \\ 0 & 1 & 3/4 \\ 0 & 0 & -7/8 \end{array}\right)$$

Now turn back into a system.

$$\boxed{\begin{aligned} x_1 - 3/2\ x_2 &= -1 \\ x_2 &= 3/4 \\ 0 &= -7/8 \end{aligned}}$$

Since we have

$0 = -7/8$

this system has $\boxed{\text{no solutions}}$

it is inconsistent.

② (b)

make into a 1

make these zeros

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -5 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right)$$

$$\xrightarrow[\substack{-5R_1+R_2 \to R_2 \\ -3R_1+R_3 \to R_3 \\ 6R_1+R_4 \to R_4}]{} \left(\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -5 \\ 0 & -1/3 & 11/3 & 25 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

make this into a 1

$$\xrightarrow{-3R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -5 \\ 0 & 1 & -11 & -75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

make these zeros

$$\xrightarrow{R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -5 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

make this into a 1

$$\xrightarrow{-\frac{1}{7}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -5 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Write out the reduced system:

$$\boxed{\begin{aligned} x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3 &= -5 \\ x_2 - 11x_3 &= -75 \\ x_3 &= 7 \end{aligned}}$$

x_1, x_2, x_3 are leading variables.
There are no free variables

$\Downarrow$

$$\boxed{\begin{aligned} x_1 &= -5 - \tfrac{2}{3}x_2 + \tfrac{1}{3}x_3 && (1) \\ x_2 &= -75 + 11x_3 && (2) \\ x_3 &= 7 && (3) \end{aligned}}$$

(3) gives $x_3 = 7$.
Now plug into (2) to get
$x_2 = -75 + 11(7)$
$= 2$
Now plug into (1) to get
$x_1 = -5 - \frac{2}{3}(2) + \frac{1}{3}(7)$
$= -4$

<u>Answer</u>: $x_1 = -4$, $x_2 = 2$, $x_3 = 7$

② (c)

put a 1 here

make these zeros

$$\left(\begin{array}{cc|c} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & -6 \end{array}\right) \xrightarrow{\frac{1}{4}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & -6 & 9 \\ -2 & 4 & -6 \end{array}\right)$$

$$\xrightarrow[2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Write down reduced system:

$$\boxed{\begin{aligned} x_1 - 2x_2 &= 3 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

x_1 is a leading variable
x_2 is a free variable

$$\boxed{x_1 = 3 + 2x_2} \quad ①$$

Set free variables:
$x_2 = t$

Solve ①:
$x_1 = 3 + 2t$

Answer:

$x_1 = 3 + 2t$
$x_2 = t$
where t is any real number

2(d)

make this into a 1

$$\left(\begin{array}{cccc|c} 0 & 10 & -4 & 1 & 1 \\ 1 & 4 & -1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 5 \\ -2 & -8 & 2 & -2 & -4 \\ 1 & -6 & 3 & 0 & 1 \end{array}\right)$$

make these into zeros

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 3 & 2 & 1 & 2 & 5 \\ -2 & -8 & 2 & -2 & -4 \\ 1 & -6 & 3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[-R_1 + R_5 \to R_5]{-3R_1 + R_3 \to R_3,\ 2R_1 + R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & -10 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 4 & -1 & -1 \end{array}\right)$$

notice that R_3 is $-R_2$ and R_5 is R_2 so an easy way to simplify is as follows

$$\xrightarrow[R_2 + R_5 \to R_5]{R_2 + R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

now make this a 1

$$\xrightarrow{\frac{1}{10}R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & 4 & -1 & 1 & 2 \\ 0 & 1 & -4/10 & 1/10 & 1/10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

reduced system is:

$$\boxed{\begin{aligned} x + 4y - z + w &= 2 \\ y - \tfrac{4}{10}z + \tfrac{1}{10}w &= \tfrac{1}{10} \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

x & y are leading variables

z & w are free variables

$\Downarrow$

$$\boxed{\begin{aligned} x &= 2 - 4y + z - w & (1) \\ y &= \tfrac{1}{10} + \tfrac{4}{10}z - \tfrac{1}{10}w & (2) \end{aligned}}$$

$\Rightarrow$

set free variables:

$z = t$

$w = s$

plug into (2):

$y = \frac{1}{10} + \frac{4}{10}t - \frac{1}{10}s$

plug into (1):

$x = 2 - 4\left(\frac{1}{10} + \frac{4}{10}t - \frac{1}{10}s\right) + t - s$

$= \frac{8}{5} - \frac{3}{5}t - \frac{3}{5}s$

$\Leftarrow$

Answer:

$x = \frac{8}{5} - \frac{3}{5}t - \frac{3}{5}s$

$y = \frac{1}{10} + \frac{4}{10}t - \frac{1}{10}s$

$z = t$

$w = s$

where t & s can be any real

#5

3 (a)

make this a 1

$$\left(\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array}\right) \xrightarrow{\frac{1}{5}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ -2 & 1 & 3 & 1 \end{array}\right)$$

make this a zero

$$\xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ 0 & \frac{1}{5} & \frac{27}{5} & 1 \end{array}\right)$$

make this a 1

$$\xrightarrow{5R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ 0 & 1 & 27 & 5 \end{array}\right)$$

reduced system is:

$$\boxed{\begin{aligned} x_1 - \tfrac{2}{5}x_2 + \tfrac{6}{5}x_3 &= 0 \\ x_2 + 27x_3 &= 5 \end{aligned}}$$

leading variables are x_1 & x_2

free variables are x_3

$$\boxed{\begin{aligned} x_1 &= \frac{2}{5}x_2 - \frac{6}{5}x_3 \quad (1) \\ x_2 &= 5 - 27x_3 \quad (2) \end{aligned}}$$

Set free variables:

$x_3 = t$

plug into (2):

$x_2 = 5 - 27t$

plug into (3):

$x_1 = \frac{2}{5}(5 - 27t) - \frac{6}{5}(t) = 2 - 12t$

Answer:

$x_1 = 2 - 12t$

$x_2 = 5 - 27t$

$x_3 = t$

where t can be any real #

3 (b)

we already have a 1 here

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{array}\right)$$

make these into zeros

make this a 1

$$\xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2\to R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & -10 & -12 & -12 & 4 \end{array}\right)$$

make this a zero

$$\xrightarrow{10R_2+R_3\to R_3} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & 1 \\ 0 & 1 & 6/5 & 6/5 & 1/5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right)$$

Write down the reduced system!

$$x_1 - 2x_2 + x_3 - 4x_4 = 1$$
$$x_2 + \frac{6}{5}x_3 + \frac{6}{5}x_3 = \frac{1}{5}$$
$$0 = 6$$

Because we have $0 = 6$ there are no solutions to the system. It is inconsistent.

④ (a)

put a 1 here

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

put zeros here

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

make this a 1

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

make this a 0

$$\xrightarrow{-R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

make this a 1

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

write down the reduced system:

$$\boxed{\begin{aligned} x_1 + 2x_2 \quad &= 0 \\ x_2 - x_3 &= 0 \\ x_3 &= 0 \end{aligned}}$$

leading variables are x_1, x_2, x_3

there are no free variables

$$\begin{aligned} x_1 &= -2x_2 \quad (1) \\ x_2 &= x_3 \quad (2) \\ x_3 &= 0 \quad (3) \end{aligned}$$

(3) gives $x_3 = 0$

plug into (2) to get

$x_2 = x_3 = 0$

plug into (1) to get

$x_1 = -2(0) = 0.$

Answer: $x_1 = 0$

$x_2 = 0$

$x_3 = 0$

4 (b)

make this a 1

$$\left(\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right)$$

make this a zero

$$\xrightarrow{\frac{1}{3}R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 1/3 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{-5R_1 + R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & -8/3 & -2/3 & -8/3 & 0 \end{array}\right)$$

make this a 1

$$\xrightarrow{-\frac{3}{8}R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 1/4 & 1 & 0 \end{array}\right)$$

Write down the reduced system:

$$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 = 0$$
$$x_2 + \frac{1}{4}x_3 + x_4 = 0$$

leading variables are x_1, x_2

free variables are x_3, x_4

$$x_1 = -\frac{1}{3}x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \quad (1)$$
$$x_2 = -\frac{1}{4}x_3 - x_4 \quad (2)$$

assign free variables:

$x_3 = s$

$x_4 = t$

plug into (2):

$$x_2 = -\frac{1}{4}s - t$$

plug into (1):

$$x_1 = -\frac{1}{3}\left(-\frac{1}{4}s - t\right) - \frac{1}{3}s - \frac{1}{3}t$$
$$= -\frac{5}{12}s$$

Answer:

$x_1 = \frac{-5}{12}s$

$x_2 = -\frac{1}{4}s - t$

$x_3 = s$

$x_4 = t$

where s & t can be any real numbers

④ (c)

put a 1 here

$$\left(\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right)$$

Make these zeros

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right)$$

make this a 1

$$\xrightarrow[2R_1 + R_4 \to R_4]{-2R_1 + R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & -8 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 2 & 2 & 4 & 0 \end{array}\right)$$

make these zeros

$$\begin{array}{l} -3R_2 + R_3 \to R_3 \\ -2R_2 + R_4 \to R_4 \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & -8 & 0 \\ 0 & 0 & 0 & 31 & 0 \\ 0 & 0 & 0 & 20 & 0 \end{array}\right)$$

make this a 1

$$\frac{1}{31} R_3 \to R_3 \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 20 & 0 \end{array}\right)$$

make this a zero

$$-20R_3 + R_4 \to R_4 \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

reduced system is:

$$\begin{aligned} w \quad - y - 3z &= 0 \\ x + y - 8z &= 0 \\ z &= 0 \end{aligned}$$

leading variables: w, x, z

free variable: y

$$\begin{aligned} w &= y + 3z \quad (1) \\ x &= -y + 8z \quad (2) \\ z &= 0 \quad (3) \end{aligned}$$

Assign free variable:

$y = t$

plug into (3):

$z = 0$

plug into (2):

$x = -y + 8z = -t$

plug into (1):

$w = y + 3z = t$

Answer:

$w = t$

$x = -t$

$y = t$

$z = 0$

t can be any real number